MEI STRUCTURED MATHEMATICS   4753/1

Methods for Advanced Mathematics (C3)

TIME   1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is 72.

This question paper consists of 6 printed pages and 2 blank pages.

Section A (36 marks)

1   Given that $y = (1+6x)^{\frac{1}{3}}$, show that $\dfrac{dy}{dx} = \dfrac{2}{y^2}$.   [4]

2   A population is P million at time t years. P is modelled by the equation

$$P = 5 + ae^{-bt},$$

where a and b are constants.

The population is initially 8 million, and declines to 6 million after 1 year.

   (i) Use this information to calculate the values of a and b, giving b correct to 3 significant figures.   [5]

   (ii) What is the long-term population predicted by the model?   [1]

3   **(i)** Express $2\ln x + \ln 3$ as a single logarithm.   [2]

   (ii) Hence, given that x satisfies the equation

$$2\ln x + \ln 3 = \ln (5x + 2),$$

   show that x is a root of the quadratic equation $3x^2 - 5x - 2 = 0$.   [2]

   (iii) Solve this quadratic equation, explaining why only one root is a valid solution of

$$2\ln x + \ln 3 = \ln (5x + 2).$$ [3]

4 Fig. 4 shows a cone. The angle between the axis and the slant edge is 30°. Water is poured into the cone at a constant rate of $2\,\text{cm}^3$ per second. At time t seconds, the radius of the water surface is $r\,\text{cm}$ and the volume of water in the cone is $V\,\text{cm}^3$.
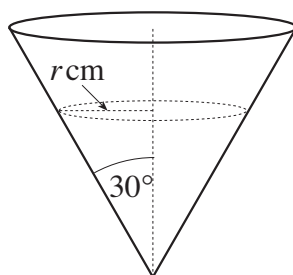

Fig. 4

   (i) Write down the value of $\dfrac{\mathrm{d}V}{\mathrm{d}t}$. [1]

   (ii) Show that $V = \dfrac{\sqrt{3}}{3}\pi r^3$, and find $\dfrac{\mathrm{d}V}{\mathrm{d}r}$. [3]

     [You may assume that the volume of a cone of height h and radius r is $\frac{1}{3}\pi r^2 h$.]

   (iii) Use the results of parts **(i)** and **(ii)** to find the value of $\dfrac{\mathrm{d}r}{\mathrm{d}t}$ when $r = 2$. [3]

5 A curve is defined implicitly by the equation

$$y^3 = 2xy + x^2.$$

   (i) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2(x+y)}{3y^2 - 2x}$. [4]

   (ii) Hence write down $\dfrac{\mathrm{d}x}{\mathrm{d}y}$ in terms of x and y. [1]

[Turn over

6 The function $f(x)$ is defined by $f(x) = 1 + 2\sin x$ for $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi$.

(i) Show that $f^{-1}(x) = \arcsin\left(\dfrac{x-1}{2}\right)$ and state the domain of this function. [4]

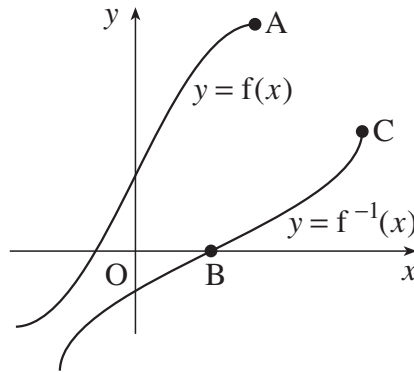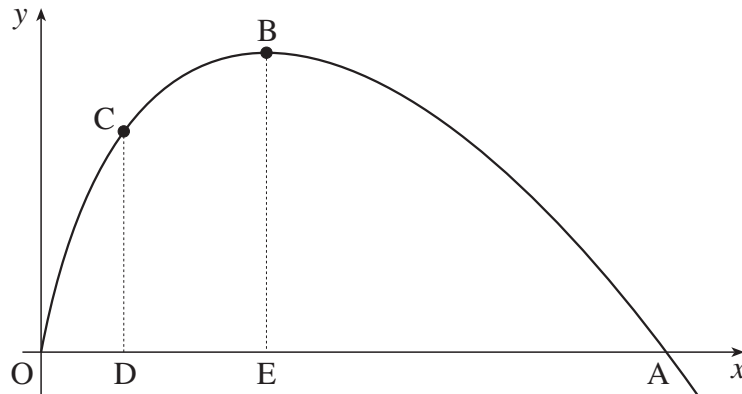Fig. 6 shows a sketch of the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

Fig. 6

(ii) Write down the coordinates of the points A, B and C. [3]

Section B (36 marks)

7 Fig. 7 shows the curve

$$y = 2x - x \ln x, \text{ where } x > 0.$$

The curve crosses the *x*-axis at A, and has a turning point at B. The point C on the curve has *x*-coordinate 1. Lines CD and BE are drawn parallel to the *y*-axis.

Not to scale

Fig. 7

   (i) Find the *x*-coordinate of A, giving your answer in terms of e. [2]

  (ii) Find the exact coordinates of B. [6]

 (iii) Show that the tangents at A and C are perpendicular to each other. [3]

  (iv) Using integration by parts, show that

$$\int x \ln x \, dx = \tfrac{1}{2}x^2 \ln x - \tfrac{1}{4}x^2 + c.$$

Hence find the exact area of the region enclosed by the curve, the *x*-axis and the lines CD and BE. [7]

[Question 8 is printed overleaf.]

8 The function $f(x) = \dfrac{\sin x}{2 - \cos x}$ has domain $-\pi \leqslant x \leqslant \pi$.

Fig. 8 shows the graph of $y = f(x)$ for $0 \leqslant x \leqslant \pi$.
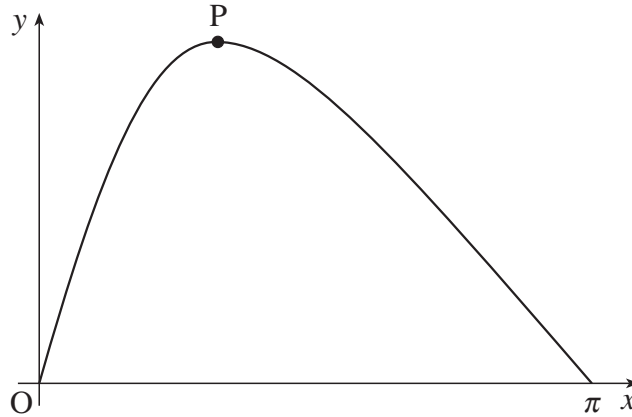

Fig. 8

(i) Find $f(-x)$ in terms of $f(x)$. Hence sketch the graph of $y = f(x)$ for the complete domain $-\pi \leqslant x \leqslant \pi$. [3]

(ii) Show that $f'(x) = \dfrac{2\cos x - 1}{(2 - \cos x)^2}$. Hence find the exact coordinates of the turning point P.

State the range of the function $f(x)$, giving your answer exactly. [8]

(iii) Using the substitution $u = 2 - \cos x$ or otherwise, find the exact value of $\displaystyle\int_0^\pi \dfrac{\sin x}{2 - \cos x}\, dx$.

[4]

(iv) Sketch the graph of $y = f(2x)$. [1]

(v) Using your answers to parts (iii) and (iv), write down the exact value of $\displaystyle\int_0^{\frac{1}{2}\pi} \dfrac{\sin 2x}{2 - \cos 2x}\, dx$.
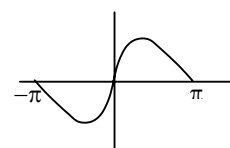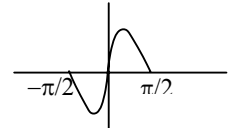
[2]

Mark Scheme 4753
January 2006

Section A

1 $y = (1 + 6x)^{1/3}$ $\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{3}(1+6x)^{-2/3}.6$ $\quad = 2(1 + 6x)^{-2/3}$ $\quad = 2[(1 + 6x)^{1/3}]^{-2}$ $\quad = \dfrac{2}{y^2}$ *	M1 B1 A1 E1	Chain rule $\dfrac{1}{3}(1+6x)^{-2/3}$ or $\dfrac{1}{3}u^{-2/3}$ any correct expression for the derivative www
or $y^3 = 1 + 6x$ $\Rightarrow x = (y^3 - 1)/6$ $\Rightarrow dx/dy = 3y^2/6 = y^2/2$ $\Rightarrow dy/dx = 1/(dx/dy) = 2/y^2$ *	M1 A1 B1 E1	Finding x in terms of y $y^2/2$ o.e.
or $y^3 = 1 + 6x$ $\Rightarrow 3y^2\, dy/dx = 6$ $\Rightarrow dy/dx = 6/3y^2 = 2/y^2$ *	M1 A1 A1 E1 [4]	together with attempt to differentiate implicitly $3y^2\, dy/dx$ $= 6$
2 (i) When $t = 0$, $P = 5 + a = 8$ $\Rightarrow a = 3$ When $t = 1$, $5 + 3e^{-b} = 6$ $\Rightarrow e^{-b} = 1/3$ $\Rightarrow -b = \ln 1/3$ $\Rightarrow b = \ln 3 = 1.10$ (3 s.f.) **(ii)** 5 million	M1 A1 M1 M1 A1ft B1 [6]	substituting $t = 0$ into equation Forming equation using their a Taking lns on correct re-arrangement (ft their a) or $P = 5$
3 (i) $\ln (3x^2)$ **(ii)** $\ln 3x^2 = \ln(5x + 2)$ $\Rightarrow 3x^2 = 5x + 2$ $\Rightarrow 3x^2 - 5x - 2 = 0$* **(iii)** $(3x + 1)(x - 2) = 0$ $\Rightarrow x = -1/3$ or 2 $x = -1/3$ is not valid as $\ln (-1/3)$ is not defined	B1 B1 M1 E1 M1 A1cao B1ft [7]	$2\ln x = \ln x^2$ $\ln x^2 + \ln 3 = \ln 3x^2$ Anti-logging Factorising or quadratic formula ft on one positive and one negative root

4 (i) $\frac{dV}{dt} = 2$	B1	
(ii) $\tan 30 = 1/\sqrt{3}$ $\qquad = r/h$		
$\Rightarrow \quad h = \sqrt{3}\, r$	M1	Correct relationship between r and h in any form From exact working only
$\Rightarrow \quad V = \frac{1}{3}\pi r^2 . \sqrt{3}r = \frac{\sqrt{3}}{3}\pi r^3$ *	E1	
$\qquad \frac{dV}{dr} = \sqrt{3}\pi r^2$	B1	o.e. e.g. $(3\sqrt{3}/3)\pi\, r^2$
(iii) When $r = 2$, $dV/dr = 4\sqrt{3}\pi$		
$\qquad \frac{dV}{dt} = \frac{dV}{dr}.\frac{dr}{dt}$		or $\frac{dr}{dt} = \frac{dr}{dV}.\frac{dV}{dt}$
$\Rightarrow \quad 2 = 4\sqrt{3}\pi\ dr/dt$	M1	substituting 2 for dV/dt and $r = 2$ into their dV/dr
	M1	
$\Rightarrow \quad dr/dt = 1/(2\sqrt{3}\pi)$ $\qquad$ or 0.092 cm s^{-1}	A1cao	
	[7]	
5(i) $y^3 = 2xy + x^2$		
$\Rightarrow \quad 3y^2 \frac{dy}{dx} = 2x\frac{dy}{dx} + 2y + 2x$	B1	$3y^2 \frac{dy}{dx} =$
$\Rightarrow \quad (3y^2 - 2x)\frac{dy}{dx} = 2y + 2x$	B1	$2x\frac{dy}{dx} + 2y + 2x$
	M1	collecting dy/dx terms on one side
$\Rightarrow \quad \frac{dy}{dx} = \frac{2(x+y)}{3y^2 - 2x}$ *	E1	www
(ii) $\frac{dx}{dy} = \frac{3y^2 - 2x}{2(x+y)}$	B1cao	
	[5]	
6(i) $y = 1 + 2\sin x$ $\ y \leftrightarrow x$	M1	Attempt to invert
$\Rightarrow \quad x = 1 + 2\sin y$		
$\Rightarrow \quad x - 1 = 2\sin y$		
$\Rightarrow \quad (x - 1)/2 = \sin y$	A1	
$\Rightarrow \quad y = \arcsin(\frac{x-1}{2})$ *	E1	
Domain is $-1 \le x \le 3$	B1	
(ii) A is $(\pi/2, 3)$	B1cao	
$\quad$ B is $(1, 0)$	B1cao	Allow $\pi/2 = 1.57$or better
$\quad$ C is $(3, \pi/2)$	B1ft	ft on their A
	[7]	

Section B

7(i) $2x - x \ln x = 0$ $\Rightarrow$ $x(2 - \ln x) = 0$ $\Rightarrow$ $(x = 0)$ or $\ln x = 2$ $\Rightarrow$ at A, $x = e^2$	M1 A1 [2]	Equating to zero
(ii) $\dfrac{dy}{dx} = 2 - x.\dfrac{1}{x} - \ln x.1$ $\qquad = 1 - \ln x$ $\dfrac{dy}{dx} = 0 \Rightarrow 1 - \ln x = 0$ $\Rightarrow$ $\ln x = 1, x = e$ When $x = e$, $y = 2e - e \ln e = e$ So B is (e, e)	M1 B1 A1 M1 A1cao B1ft [6]	Product rule for $x \ln x$ d/dx (ln x) = 1/x 1 − ln x o.e. equating their derivative to zero $x = e$ $y = e$
(iii) At A, $\dfrac{dy}{dx} = 1 - \ln e^2 = 1 - 2$ $\qquad\qquad = -1$ At C, $\dfrac{dy}{dx} = 1 - \ln 1 = 1$ $1 \times -1 = -1 \Rightarrow$ tangents are perpendicular	M1 A1cao E1 [3]	Substituting x=1 or their e^2 into their derivative -1 and 1 www
(iv) Let $u = \ln x$, dv/d$x = x$ $\Rightarrow v = \frac{1}{2} x^2$ $\int x \ln x dx = \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x^2.\frac{1}{x}dx$ $= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x dx$ $= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + c$ * $A = \int_1^e (2x - x \ln x)dx$ $= \left[x^2 - \frac{1}{2}x^2 \ln x + \frac{1}{4}x^2 \right]_1^e$ $= (e^2 - \frac{1}{2}e^2 \ln e + \frac{1}{4} e^2) - (1 - \frac{1}{2}1^2 \ln 1 + \frac{1}{4} 1^2)$ $= \frac{3}{4}e^2 - \frac{5}{4}$	M1 A1 E1 B1 B1 M1 A1 cao [7]	Parts: $u = \ln x$, dv/d$x = x \Rightarrow v = \frac{1}{2} x^2$ correct integral and limits $\left[x^2 - \frac{1}{2}x^2 \ln x + \frac{1}{4}x^2 \right]$ o.e. substituting limits correctly

8 (i) $f(-x) = \dfrac{\sin(-x)}{2-\cos(-x)}$ $= \dfrac{-\sin(x)}{2-\cos(x)}$ $= -f(x)$	M1 A1 B1 [3]	substituting $-x$ for x in f(x) Graph completed with rotational symmetry about O.
(ii) $f'(x) = \dfrac{(2-\cos x)\cos x - \sin x.\sin x}{(2-\cos x)^2}$ $= \dfrac{2\cos x - \cos^2 x - \sin^2 x}{(2-\cos x)^2}$ $= \dfrac{2\cos x - 1}{(2-\cos x)^2}$ * $f'(x) = 0$ when $2\cos x - 1 = 0$ $\Rightarrow \cos x = \frac{1}{2}$, $x = \pi/3$ When $x = \pi/3$, $y = \dfrac{\sin(\pi/3)}{2-\cos(\pi/3)} = \dfrac{\sqrt{3}/2}{2-1/2}$ $= \dfrac{\sqrt{3}}{3}$ So range is $-\dfrac{\sqrt{3}}{3} \le y \le \dfrac{\sqrt{3}}{3}$	M1 A1 E1 M1 A1 M1 A1 B1ft [8]	Quotient or product rule consistent with their derivatives Correct expression numerator = 0 Substituting their $\pi/3$ into y o.e. but exact ft their $\dfrac{\sqrt{3}}{3}$
(iii) $\displaystyle\int_0^\pi \dfrac{\sin x}{2-\cos x}dx$ let $u = 2 - \cos x$ $\Rightarrow du/dx = \sin x$ When $x = 0$, $u = 1$; when $x = \pi$, $u = 3$ $= \displaystyle\int_1^3 \dfrac{1}{u}du$ $= \left[\ln u\right]_1^3$ $= \ln 3 - \ln 1 = \ln 3$ - *or* $= \left[\ln(2-\cos x)\right]_0^\pi$ $= \ln 3 - \ln 1 = \ln 3$	M1 B1 A1ft A1cao M2 A1 A1 cao [4]	$\displaystyle\int \dfrac{1}{u}du$ $u = 1$ to 3 $[\ln u]$ $[k\ln(2-\cos x)]$ $k = 1$
(iv)	B1ft [1]	Graph showing evidence of stretch s.f. ½ in x – direction
(v) Area is stretched with scale factor ½ So area is ½ ln 3	M1 A1ft [2]	soi ½ their ln 3

General Comments

This paper was perhaps a little tougher than last summer's, and attracted a wide range of responses from candidates. Weaker candidates found it a tough test, and showed evidence of a lack of maturity in the understanding of the concepts and in problem solving and algebraic ability – it should be born in mind that this paper, unlike P2, is of full A2 standard. However, average candidates found plenty to do, and there were many excellent scripts achieving over 60 marks.

Although most candidates managed to attempt all the questions, there was some evidence of them rushing in question 8. This was usually caused by inefficient methods, or repeating the same question a number of times.

The general level of algebra remains a source of concern. In particular, many candidates are omitting brackets, and this led to marks being lost through algebraic errors with negative signs – see the comments on questions 7(ii), (iv) and 8(ii). A large number of candidates also failed to give their answers exactly in questions 7 and 8.

Comments on Individual Questions

Section A

1.   The chain rule was generally well known, although there were occasional errors with the derivative of $u^{1/3}$. Weaker candidates failed to show the result was $1/y^2$; stronger ones cubed both sides and used implicit differentiation, which proves the result very efficiently.

2.   Many candidates misinterpreted the units of P and used 8 000 000 and 6 000 000 instead of 8 and 6. We allowed generous follow through for this, so that they could achieve 5 marks out of 6 with this error. There were the usual errors in taking logs to solve for b, but in general this was quite well done. The final part was quite often omitted.

3.   Although able candidates sailed through this test, solving a logarithmic equation like this seemed to be a fairly unfamiliar context to many candidates. The answer $2 \ln 3x$ was a common error in part (i), and part (ii) suffered accordingly. However, there were 2 easy marks in (iii) for solving the quadratic equation, preferably by factorisation. We allowed follow through on negative roots for the final part, in which candidates had to phrase their answers carefully: for example, 'logarithms cannot be negative' was not allowed here.

4.   This question proved to be the most demanding of the section A questions. In particular, the derivation of the volume of the cone result was done successfully by only the best candidates. Although only awarded 2 marks, this may have put off weaker candidates, although most wrote down the dV/dt for a mark in part (i), and recognised the chain rule in part (iii).

5.   Having the result of the implicit differentiation proved helpful to candidates here, and most either achieved 4 or 5 marks or 1 or 0. The derivative of the xy term stumped some candidates, and although dx/dy = 1/(dy/dx) was well known, we wanted to see the resulting fraction inverted for the final mark.

6 The first three marks proved accessible to most candidates, provided they understood the notation and concept of inverse trigonometric functions. Candidates who started $x = 1 + 2 \sin x$ were penalised, however, and the domain was poorly answered. The use of radians in degrees in the coordinates of A and C was quite a common error, and quite a few candidates spent a lot of time trying to establish the coordinates, for example using calculus.

Section B
7 Plenty of candidates scored well on this question. However, the area in part (iv) lost three marks to virtually all and sundry.
  (i) Most set y to zero, but many failed to solve the subsequent equation – again, equations, even simple ones, involving logarithms appeared unfamiliar to many candidates.
  (ii) Most knew the derivative of $\ln x$, and could apply a product rule to $x \ln x$. However, omitting the bracket led quite a few to a derivative of
  $2 - 1 + \ln x = 1 + \ln x$, and subsequent errors in the coordinates of the turning point. Many candidates worked approximately here, rather than expressing their answers in terms of e.
  (iii) These 3 marks relied upon part (ii) being correct, although we allowed 1 follow through method mark on their derivative in part (ii). The product of the gradients result was well known.
  (iv) The integration by parts was generally well done, although some took u and v the wrong way round, and we required to see the integral term simplified to $\frac{1}{2} x$ before being integrated. However, a large majority then used the resultant integrand for the area, rather than integrating $2x - x \ln x$! This lost three marks to virtually all − the final answer was very rarely seen.

8 Some candidates were clearly pushed for time to answer this question.
  (i) The results $\sin (-x) = -\sin x$ and $\cos (-x) = \cos x$ were not well known, and so answers to this part were weak. Many thought the function was even; but an equal number or more guessed it was odd without being able to prove it, and completed the graph correctly.
  (ii) The quotient rule was well known, although some made errors in the derivative. Missing brackets out in the numerator often led to errors in showing the result in the paper, for example using $\cos^2 x - \sin^2 x = 1$. Many candidates found $x = 60$ rather than $\pi/3$, and few found the exact value of the y−coordinate of the turning point, or the range.
  (iii) Most good candidates got four marks here quite easily. Weaker candidates failed to use substitution correctly in this trigonometric context.
  (iv) We followed through their graphs in part (i), and as a result many achieved this mark – the stretch was well known, although some used scale factor 2 rather than $\frac{1}{2}$.
  (v) Again, we allowed generous follow through on answers to part (iii) – if this was halved, they got the marks. Some saw the point about the stretch, many re-started the integral (and often ran out of time!).